Mathematics of the MML functional quantizer modules for VCV Rack software synthesizer

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Abstract

We detail the mathematical formulation of the line of "functional quantizer" modules developed by the Mathematics and Music Lab (MML) at Michigan Technological University, for the VCV Rack software modular synthesizer platform, which allow synthesizer players to tune oscillators to new musical scales based on mathematical functions. For example, we describe the recently-released MML Logarithmic Quantizer (LOG QNT) module that tunes synthesizer oscillators to the non-Pythagorean musical scale introduced by pop band The Apples in Stereo.

The Mathematics and Music Lab (MML) at Michigan Technological University is an interdisciplinary working group in the departments of Visual and Performing Arts (VPA) and Mathematical Sciences, sponsored by professors M. G. Maxwell (VPA) and R. Schneider (Math. Sci.), with student, faculty and industry collaborators. The goals of MML are to produce futuristic works of music and installation art, to explore and extend music theory using ideas from mathematics, and to invent new audio hardware and software to bring those projects to life.

In 2007, an unusual musical scale tuned to the logarithms of positive integers was introduced on the album *New Magnetic Wonder* by indie pop band The Apples in Stereo [1]. This scale was invented by the fifth author (R. Schneider), and was subsequently studied in the papers [2], [3], and featured on the 2022 instrumental album *Songs for Other Worlds* by Robert Schneider [4]. It was named the "non-Pythagorean" scale because the frequencies defining the tones of the scale are irrational numbers bearing little relation to the rational sequence of frequencies defining the traditional chromatic scale; the discovery of the formula for the chromatic scale is attributed to Pythagoras. By this definition, many scales used in world musical traditions (e.g. [5]) and experimental music [6], [7] qualify as non-Pythagorean; here we focus on a class of "functionally quantized" scales based on mathematical functions with certain amenable properties.

MML has developed a line of *functional quantizer* modules for the widely-used VCV Rack software synthesizer platform [8], coded in C++ by MTU Mathematical Sciences master's student C. McCarthy, based on mathematical formulas derived chiefly by M. Schneider, then a Mathematics and Computer Science double major at University of Georgia. The MML modules allow synthesizer players to play new musical scales tuned to mathematical functions. Our first functional quantizer module, the MML Logarithmic Quantizer (LOG QNT) [9] released in March 2024, modulates control voltage (CV) signals to produce the non-Pythagorean scale in voltage controlled oscillators; see Fig. 1. *The MML functional quantizer development team is:* C. McCarthy (lead programmer), M. G. Maxwell and R. Schneider (project design), M. Schneider and A. V. Sills (mathematical proofs, additional programming), J. Pfeffer (graphics).

Here we describe the mathematical background of the MML functional quantizer line of VCV Rack modules, that make infinite families of mathematical musical scales accessible to modular synthesists. Generalizing the non-Pythagorean scale of the fifth author, a twelve-tone-per-octave scale, let us define a *functionally quantized (FQ) musical scale with T tones per octave* as follows. For $0 \leq x \leq 1$, let $f(x)$ be a strictly increasing function such that $f(0) = 1,\ f(1) = 2$, and let F_0 denote an arbitrary *base frequency* in hertz (Hz) that serves as the root note in the scale. If we identify musical pitch with the corresponding frequency (in hertz), then the nth pitch F_n of the first octave of the corresponding FQ scale is defined by the relation

$$F_n = F_0 \cdot f(n/T), \qquad n = 0, 1, 2, 3, \ldots, T-1, T. \tag{1}$$

The sequence of pitches is repeated in higher/lower octaves by doubling/halving the frequencies.

Twelve tone equal temperament in music theory is the prototype for FQ scales: set $f(x) = 2^x$, $T = 12$, such that $F_n = F_0 \cdot 2^{n/12}$. In the logarithmic non-Pythagorean musical scale defined in [2], the fifth author uses $T = 12$ tones playable on a piano keyboard, along with the function

$$f(x) = \frac{1}{2}\log_2(4+12x), \qquad x = 0, \frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \ldots, \frac{n}{12}, \ldots, \frac{11}{12}, 1, \tag{2}$$

where $\log_2 t,\ t > 0$, is the base-2 logarithm function, noting $f(x)$ satisfies the boundary conditions $f(0) = 1, f(1) = 2$ above. Note that we can use the *floor function* $\lfloor t \rfloor, t \in \mathbb{R}$, to rewrite

$$f(x) = \frac{1}{2}\log_2(4 + \lfloor 12x \rfloor), \quad x \in [0,1], \tag{3}$$

since $\lfloor Tx \rfloor = 0, 1, 2, \ldots, T-1, T$, as x attains the respective values $x = 0, \frac{1}{T}, \frac{2}{T}, \frac{3}{T}, \ldots, \frac{T-1}{T}, 1$, when x moves continuously to the right on the interval $[0,1]$, which the reader can check.

Now, in common volt-per-octave CV input calibration for modular synthesizers — or, more generally, a V_{ref}-volts-per-octave calibration, $V_{\text{ref}} > 0$ volts, where V_{ref} is a *reference voltage* — let us notate $V_{\text{in}} \geq 0$ volts for the *input voltage* and $V_{\text{out}} \geq 0$ volts for the corresponding *functionally quantized output voltage* that our LOG QNT module for VCV Rack produces. Set $V_{\text{ref}} = 1$ volt in usual volt-per-octave calibration; Buchla synths use $V_{\text{ref}} = 1.2$ volts. So we have

$$F_n = F_0 \cdot 2^{V_{\text{out}}/V_{\text{ref}}}, \qquad n = 0, 1, 2, 3, \ldots, \tag{4}$$

where $V_{\text{out}} = V_{\text{out}}(n)$ is the sequence of FQ control voltages yielding the sequence of pitches F_n.

Define the *fractional part function* $\text{frac}(t) := t - \lfloor t \rfloor,\ t \in \mathbb{R}$. Then the "$f(x)$-quantized" output voltage for a T-tone scale is given by the formula

$$V_{\text{out}} \;=\; V_{\text{ref}} \cdot \left\{ \lfloor V_{\text{in}}/V_{\text{ref}} \rfloor + \log_2 f\left(\lfloor T\,\text{frac}\left(V_{\text{in}}/V_{\text{ref}} \right) \rfloor \right) \right\}, \tag{5}$$

as V_{in} increases, which is obtained by comparing equations (1) and (4), taking base-2 logarithms and doing a little algebra to solve for V_{out}. This simplifies in the usual volt-per-octave case to

$$V_{\text{out}} \;=\; \lfloor V_{\text{in}} \rfloor + \log_2 f\left(\lfloor T\,\text{frac}(V_{\text{in}}) \rfloor \right) \ \text{volts}. \tag{6}$$

For the logarithmic non-Pythagorean scale introduced in [1], [2], the FQ output voltage is computed by setting $x = \text{frac}(V_{\text{in}}/V_{\text{ref}}),\ T = 12$, and $f(x)$ as in (3) to solve for V_{out}, yielding

$$V_{\text{out}} \;=\; V_{\text{ref}} \cdot \left\{ \lfloor V_{\text{in}}/V_{\text{ref}} \rfloor - 1 + \log_2 \log_2 \left(4 + \lfloor 12\,\text{frac}\left(V_{\text{in}}/V_{\text{ref}} \right) \rfloor \right) \right\} \tag{7}$$

for $V_{\text{in}} \geq 0$ volts. In the usual volt-per-octave case, equation (7) simplifies to

$$V_{\text{out}} \;=\; \lfloor V_{\text{in}} \rfloor - 1 + \log_2 \log_2 \left(4 + \lfloor 12\,\text{frac}(V_{\text{in}}) \rfloor \right) \ \text{volts}, \tag{8}$$

which is the formula for V_{out} used in the C++ code for the MML LOG QNT module for VCV Rack [10]. Whether V_{in} is a continuous sweep or an incrementally increasing voltage, formulas (5), (6), (7) and (8) output a sequence of FQ voltage values, computed to 6 decimal places of accuracy in single precision floating point arithmetic (as used in VCV Rack as of this writing).

Other functional quantizers MML will release for VCV Rack in 2024 (all having $T = 12$) are:

(a) Square Root Quantizer (SQT QNT) with $f(x) = \frac{1}{2}\sqrt{4+12x},\ 0 \leq x \leq 1$;
(b) Sine Quantizer (SIN QNT) with $f(x) = 1 + \sin(\pi x/2),\ 0 \leq x \leq 1$;
(c) Power Quantizer (POW QNT) with $f(x) = \frac{1}{2}\left(2^a + \frac{4^a - 2^a}{12}x\right)^{\frac{1}{a}},\ 0 \leq x \leq 1, 0 < a < \infty$;
(d) Power Quantizer 2 (POW QNT 2) with $f(x) = 1 + x^a,\ 0 \leq x \leq 1, 0 \leq a < \infty$.

The SQT QNT module was conceived by R. Schneider and A. V. Sills; the SIN QNT module was conceived by M. G. Maxwell and R. Schneider; and the two-parameter POW QNT modules were conceived by C. McCarthy. For programming details about the MML modules, see [10].

Figure 1. *MML Logarithmic Quantizer (LOG QNT) module for VCV Rack*

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